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THE CONSTRUCTION OF AN ACCURATE LOWER BOUND FOR THE REAL PARTS OF THE EIGENVALUES OF AN M-MATRIX

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The construction of an accurate lower bound for the real parts of the eigenvalues of an M-matrix *)

by

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ABSTRACT

Let A be an M-matrix, i.e. A is non-singular, real, irreducible and weakly diagonally domainant and has positive diagonal and non-positive off-diagonal elements. Via the graph of A we construct a vector W such that AW is positive. This yields a lower bound of the spectrum, which is optimal in certain problems.

KEY WORDS & PHRASES: M-*matrix, lower bound for eigenvalues*

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*) This report will be submitted for publication elsewhere.

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1. INTRODUCTION

In this note we shall derive a lower bound for the real parts of the eigenvalues of a real matrix, which is invertible, irreducible and weakly diagonally dominant with positive diagonal elements and non-positive off-diagonal elements. Such a matrix is usually called an M-matrix, cf. [1, ch. 6].

If A is such a matrix of dimension $n \times n$ $(n > 1)$, then its matrix elements satisfy

(1) $a_{ij} \leq 0$, $i,j = 1,\dots,n$ with $i \neq j$;

(2) $r_i := \Sigma_{j=1}^{n} a_{ij} \geq 0$ and $\Sigma_{j=1}^{n} a_{ji} \geq 0$, $i = 1\dots n$;

(3) $\exists k$: $r_k \neq 0$;

(4) $\forall i\ \exists j \neq i$: $a_{ij} \neq 0$.

We note that the third condition is equivalent to the invertability of A.

The problem arose in a singularly perturbed boundary value problem, in which we could reduce the approximation problem to a finite dimensional subspace by a Galerkin method, cf. [2,3]. In this finite dimensional subspace we had to find a lower bound for the bilinear form x^TAx in which A satisfies the conditions stated above and whose non-zero row-excesses and non-zero off-diagonal matrix elements are of the form $\pm\exp(p_{ij}/\varepsilon)$. By the method which we shall describe here, we were able to derive a lower bound which is asymptotically (for $\varepsilon \to +0$) of optimal order.

Without loss of generality we may assume that A is symmetric. If A is not symmetric, its eigenvalues are contained in the numerical range

$$\{x^*Ax \mid x^*x = 1,\ x \in \mathbb{C}^n\}.$$

Since A is real we have $A^T = A^*$ and

$$\mathrm{Re}\ x^*Ax = \frac{1}{2}\, x^*(A^*+A)x.$$

Hence, a lower bound for the smallest eigenvalue of the symmetric part of A is also a lower bound for A itself.

2. THE CONSTRUCTION OF A LOWER BOUND

We shall construct a positive vector W, such that AW is positive too, and derive from this vector the lower bound; a positive vector is a vector with positive components. The method is analogous to barrier function techniques for elliptic differential equations of second order.

Let G be the graph of the matrix A, which has the vertices $g_1, \ldots, g_n$. The vertices g_i and g_j are connected by an edge iff $a_{ij} \neq 0$. We shall call g_k a boundary vertex if the sum of the k-th row has a positive excess, i.e. if $r_k > 0$.

We shall call g_i and g_j neighbours of each other, if they are connected by an edge. We shall devote by v_j the valency or number of neighbours of the vertex g_j. Since A is irreducible, its graph G is connected.

We define $\mathcal{F}$ as the class of all spanning rooted trees in G, whose roots are a boundary vertex; we note that $\mathcal{F}$ contains as many copies of a spanning tree in G, as there are boundary vertices. For a given rooted tree $T \in \mathcal{F}$ all vertices located on the path in T from the root to the vertex g_i are called the ascendants of g_i in T, and all vertices for which g_i is an ascendant in T are called the descendants of g_i in T. The ascendant neighbour is called the predecessor and a descendent neighbour is called a successor in T.

For a given rooted tree $T \in \mathcal{F}$ we define recursively the functions w_T and γ_T on its vertices by:

(1) if g_k is the root of T, then

$$\gamma_T(k) := 1/(v_k+1) \quad \text{and} \quad w_T(k) := 1/r_k;$$

(2) if g_j is a successor of g_i in T, then

$$\gamma_T(j) := \gamma_T(i)/v_j$$

$$w_T(j) := w_T(i) + \gamma_T(i)/\lceil a_{ij} \rceil.$$

We define the vectors W and Γ in $\mathbb{R}^n$ in such a way, that the pair of components (W_k, Γ_k) is equal to the pair $(w_T(k), \gamma_T(k))$ for which $w_T(k)$ takes its minimal value if T ranges over the class $\mathcal{F}$. The k-th component of AW satisfies the equation

$$(AW)_k = \sum_{j=1}^{n} a_{kj}W_j = r_kW_k + \sum_{j=1,j\neq k}^{n} |a_{kj}|(W_k-W_j).$$

Let $T \in \mathcal{F}$ be such that $W_k = w_T(k)$. If g_k is not the root of T, then it has a predecessor in T, say g_p. Since W_p is minimal, we have: $W_p \leq w_T(p)$. For a neighbour g_s of g_k in G, which is not the predecessor of g_k in T, we have the following possibilities:

(1) g_s is a successor of g_k in T; hence, the minimality of W implies: $W_s \leq w_T(s)$.

(2) g_s is an ascendant of g_k in T, hence

$$W_s \leq w_T(s) \leq w_T(k) = W_k.$$

(3) g_s is neither successor nor ascendant of g_k; then another rooted tree $t \in \mathcal{F}$ exists, which has the same root as T has, in which the path from the root to g_k is the same as in T, and in which g_s is a successor of g_k. Hence

$$W_s \leq w_t(s) = w_T(k) + \gamma_T(k)/|a_{ks}|.$$

Altogether this implies

$$(AW)_k \geq \gamma_T(k) = \Gamma_k.$$

We see that both W and AW are positive vectors.

THEOREM. *The constant* Λ, *defined by*

$$\Lambda := \min_k \Gamma_k/W_k$$

is a lower bound for the smallest eigenvalue of A.

PROOF. Assume, that an eigenvector x exists with $Ax = \lambda x$ and $\lambda < \Lambda$; x has at least one positive component, otherwise we take $-x$ instead.

We consider the vector $tW - x$, in which t is chosen such that $tW_i - x_i \geq 0$, $\forall i$, and $tW_k - x_k = 0$ for some k. By the assumption $\lambda < \Lambda$ this vector satisfies the inequality

$$(A(tW-x))_k \geq \Lambda tW_k - \lambda x_k \geq 0.$$

On the other hand, $tW_k - x_k = 0$ and the non-positivity of the off-diagonal elements of A imply:

$$(A(tW-x))_k = \sum_j a_{kj}(tW_j - x_j) < 0.$$

Hence the assumption $\lambda < \Lambda$ is false. □

3. OPTIMALITY OF THE LOWER BOUND

In the case where the non-zero row-excesses and the non-zero off-diagonal elements of the matrix A are exponentials of the form $\pm\exp(p_{ij}/\varepsilon)$, the lower bound is asymptotically of the same order for $\varepsilon \to 0$ as the smallest eigenvalue is.

Since the smallest eigenvalue λ_1 is equal to the minimum of the numerical range of A,

$$\lambda_1 := \min\{x^* Ax \mid x \in \mathbb{C}^n,\ x^* x = 1\},$$

we find

$$\lambda_1 \leq W^T AW / W^T W,$$

where W is the positive vector constructed in the previous section. It is easily seen that $W^T AW$ is of the order of the largest component of W and that $W^T W$ is of the order of the square of this component, hence $W^T AW/W^T W$ is of the same order as Λ is and λ_1 is squeezed in between.

This optimality remains true if A is not symmetric, since the smallest eigenvalue of A is real and the associated eigenvector has positive components, cf. [4, ch. 13].

The algorithm presented here for computation of a lower bound, may not look very practical, since the number of trees on G can be very large. Therefore, we remark that every tree in G also yields a (possibly smaller) lower bound, namely $\max_k \gamma_T(k)/w_T(k)$.

Moreover, in the case where the matrix elements are exponentials, as stated above, it is relatively easy to find the asymptotic order of Λ as follows. Augment the graph G with as many external vertices as there are boundary vertices (or non-zero row-excesses), and connect to each boundary vertex exactly one external vertex. Denote the external vertices by $g_{n+1},\ldots,g_{n+m}$ and denote the augmented graph by $\tilde{G}$. Assign to the edge between g_i and g_j the value of $|a_{ij}|$ if g_i and g_j are in G and assign to the edge between the boundary element g_k and the external element connected with it the value of r_k. Now we skip from $\tilde{G}$ the edge with smallest value and repeat this skipping as long as in the resulting graph there remains from each vertex at least one path to an external vertex. The minimum value of all remaining edges is of the same order (for $\varepsilon \to 0$) as Λ is.

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